

The Global Stability of Two Harvesting Species with variable Effort Rate

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ABSTRACT:

Exploitation of biological resources as practiced in fishery, and forestry has strong impact on dynamic evolution of biological population. The over exploitation of resources may lead to extinction of species which adversely affects the ecosystem. However, reasonable and controlled harvesting is beneficial from economical and ecological point of view. The global dynamics of two ecologically independent species which are being harvested with variable effort have been discussed. The local and global dynamics of the mathematical model is studied. The co-existence of species in the form of stable equilibrium point is possible.

INTRODUCTION:

The research on harvesting in predator-prey systems has been of interest to economists, ecologists and natural resource management for some time now. The optimal management of renewable resources has been extensively studied by many authors. The mathematical aspects of management of renewable resources have been discussed by [10]. The problem of combined harvesting of two ecologically independent species has been studied. The effects of harvesting on the dynamics of interacting species have been studied Chaudhuri et.al. [6-9] with constant harvesting, the prey predator model is found to have interesting dynamical behavior including stability, Hopf bifurcation and limit cycle [1-10].

The multi species food web models have found to have rich dynamical behavior. S Kumar et. al. [9] have investigated the harvesting of predator species predating over two preys

THE MATHEMATICAL MODEL:

Consider two independent biological species with densities X_1 and X_2 with logistic growth. The Mathematical model of two harvesting prey species with effort rate is given by the following system of ordinary differential equations:

$$\begin{aligned} \frac{dX_1}{dT} &= r_1 X_1 \left(1 - \frac{X_1}{K_1} \right) - \frac{A_1 q_1 X_1 E}{1 + B_1 X_1 + B_2 X_2} = X_1 f_1(X_1, X_2, E) \\ \frac{dX_2}{dT} &= r_2 X_2 \left(1 - \frac{X_2}{K_2} \right) - \frac{A_2 q_2 E X_2}{1 + B_1 X_1 + B_2 X_2} = X_2 f_2(X_1, X_2, E) \\ \frac{dE}{dT} &= E (h(X_1, X_2) - C) = Ek \left(\frac{p_1 q_1 A_1 X_1 + p_2 q_2 A_2 X_2}{1 + B_1 X_1 + B_2 X_2} - C \right) = E f_3(X_1, X_2) \end{aligned} \quad (1)$$

The logistic growth is considered for the two preys. The model does not consider any direct competition between the two populations. The constants K_i, r_i, A_i , and B_i , are model parameters assuming only positive values. The effort E is applied to harvest both the species and C is total cost of fishing. The harvesting is proportional to the product of effort E and the fish population density X_i .

Let the constant M_0 is the reference value of E . Introduce the following dimensionless transformations:

$$t = r_1 T, y_i = X_i / K_i (i=1,2), x = E / M_0, w_1 = A_1 q_1 E_0 / r_1, w_2 = B_1 K_1, w_3 = B_2 K_2$$

$$w_4 = r_2 / r_1, w_5 = A_2 q_2 M / r_{10}, w_6 = k K_1 / M_0, w_7 = k K_2 / M_0;$$

The dimensionless nonlinear system is obtained as:

$$\begin{aligned} \frac{dy_1}{dt} &= y_1 \left(1 - y_1 - \frac{w_1 x}{1 + w_2 y_1 + w_3 y_2} \right) = y_1 f_1(y_1, y_2, x) \\ \frac{dy_2}{dt} &= y_2 \left((1 - y_2) w_4 - \frac{w_5 x}{1 + w_2 y_1 + w_3 y_2} \right) = y_2 f_2(y_1, y_2, x) \\ \frac{dx}{dt} &= x \left(\frac{p_1 w_1 w_6 y_1 + p_2 w_5 w_7 y_2}{1 + w_2 y_1 + w_3 y_2} - C \right) = x f_3(y_1, y_2) \end{aligned} \quad (2)$$

EXISTENCE OF EQUILIBRIUM POINTS:

THEOREM 1 The positive non-zero equilibrium E_6 of nonlinear harvesting model (2) exists provided the following conditions are satisfied:

$$C < w_1 p_1 w_6 / w_2; \quad C < w_5 p_2 w_7 / w_3 \quad (3)$$

Proof: The isoclines of harvesting model (2) are given by

$$f_1(y_1, y_2, x) = 0; \quad f_2(y_1, y_2, x) = 0; \quad f_3(y_1, y_2) = 0 \quad (4)$$

Using first of (6) we get $x^* = (1 - y_1^*)(1 + w_2 y_1^* + w_3 y_2^*) / w_1$

Solving first and second of (6) we get

$$(p_1 w_1 w_6 - w_2 C) y_1^* + (p_2 w_5 w_7 - w_3 C) y_2^* = C \quad (5)$$

Using third of (6) we get

$$y_1^* - (w_4 w_1 / w_5) y_2^* = (w_5 - w_4 w_1) / w_5 \quad (6)$$

Now solving (7) and (8) we get

$$y_1^* = \frac{(w_5 - w_1 w_4)(w_7 p_2 w_5 - C w_3) + C w_1 w_4}{w_5 (w_7 p_2 w_5 - C w_3) + w_1 w_4 (w_1 p_1 w_6 - C w_2)} \quad y_2^* = \frac{C w_5 - (w_5 - w_1 w_4)(w_1 p_1 w_6 - C w_2)}{w_5 (w_7 p_2 w_5 - C w_3) + w_1 w_4 (w_1 p_1 w_6 - C w_2)}$$

The positive non-zero biological equilibrium $E_6 = (y_1^*, y_2^*, x^*)$ exists provided the conditions (3) are satisfied.

STABILITY ANALYSIS:

The following theorem gives the conditions for the locally stability of the nonzero positive equilibrium point $E_6 = (y_1^*, y_2^*, x^*)$.

Theorem 1: The positive non-zero biological feasible equilibrium $E_6 = (y_1^*, y_2^*, x^*)$ is locally asymptotically stable if the following conditions are satisfied:

$$x^* > w_2 (1 - y_1^*)^2; \quad (7)$$

$$w_4 w_1^2 x^* > w_3 w_5 (1 - y_1^*)^2 \quad (8)$$

$$w_1^2 x^* y_1^* > (w_1 w_2 y_1^* + w_3 w_5 y_2^*) (1 - y_1^*)^2 \quad (9)$$

$$w_4 w_1^2 x^* y_2^* > (w_1 w_2 y_1^* + w_3 w_5 y_2^*) (1 - y_1^*)^2 \quad (10)$$

Proof: Assume $y_1 = y_1^* + u, y_2 = y_2^* + v, y_3 = y_3^* + w$; where u, v, w are small perturbations. The coefficients of the variational matrix about $E_6 = (y_1^*, y_2^*, x^*)$ are given by

$$\begin{aligned}
& ; \\
a_{11} &= y_1^* \left[-1 + \frac{w_1 w_2 x^*}{(1 + w_2 y_1^* + w_3 y_2^*)^2} \right]; \quad a_{12} = \frac{w_1 w_3 y_1^* x^*}{(1 + w_2 y_1^* + w_3 y_2^*)^2}; \quad a_{13} = -\frac{w_1 y_1^*}{(1 + w_2 y_1^* + w_3 y_2^*)}; \\
a_{21} &= \frac{w_2 w_5 y_2^* x^*}{(1 + w_2 y_1^* + w_3 y_2^*)^2}; \quad a_{22} = y_2^* \left[-w_4 + \frac{w_3 w_5 x^*}{(1 + w_2 y_1^* + w_3 y_2^*)^2} \right]; \quad a_{23} = -\frac{w_5 y_2^*}{(1 + w_2 y_1^* + w_3 y_2^*)}; \\
a_{31} &= \frac{[w_1 w_6 p_1 + (w_3 w_6 w_1 p_1 - w_5 w_7 w_2 p_2) y_2^*] x^*}{(1 + w_2 y_1^* + w_3 y_2^*)^2}; \\
a_{32} &= \frac{[w_7 w_5 p_2 + (w_5 w_7 w_2 p_2 - w_1 w_6 w_3 p_1) y_1^*] x^*}{(1 + w_2 y_1^* + w_3 y_2^*)^2}; \quad a_{33} = 0;
\end{aligned}$$

The corresponding characteristic equation is

$$\begin{aligned}
& \lambda^3 + a_0 \lambda^2 + a_1 \lambda + a_2 = 0 \\
& \text{with } a_0 = -(a_{11} + a_{22}), \quad a_1 = a_{11} a_{22} - a_{12} a_{21} - a_{32} a_{23} - a_{13} a_{31} \\
& \text{and } a_2 = a_{11} a_{23} a_{32} + a_{13} a_{31} a_{22} - a_{12} a_{23} a_{31} - a_{13} a_{21} a_{32}
\end{aligned} \tag{11}$$

According to Routh-Hurwitz criterion for stability, the conditions are

$$a_0 > 0, a_1 > 0, a_2 > 0 \text{ and } a_0 a_1 - a_2 > 0.$$

The necessary condition for $a_0 > 0$ gives stability conditions (7) and (8). The positiveness of a_0 ensures that $a_1 > 0, a_2 > 0$

Further the necessary condition for $a_0 a_1 - a_2 > 0$ gives the stability conditions (9) and (10).

Thus, the positive non-zero biological feasible equilibrium E_6 is locally asymptotically stable if the conditions given by (7-10) are satisfied.

The following theorem gives the conditions for the global stability of the nonzero positive equilibrium point.

Theorem 2 Let the local stability conditions given by (7-10) hold. The positive non-zero biological feasible equilibrium $E_6 = (y_1^*, y_2^*, x^*)$ is global stable if the following condition is satisfied:

$$\begin{aligned}
& (w_3 - \alpha w_2 w_4)^2 < 4\alpha w_4 ((1 + w_2 y_1^* + w_3 y_2^*) - w_2) ((1 + w_2 y_1^* + w_3 y_2^*) - w_3) \\
& \alpha = \frac{w_1 (w_7 w_5 p_2 - w_3 C)}{w_5 (w_1 w_6 p_1 - w_2 C)} > 0
\end{aligned} \tag{12}$$

Proof: Assume $y_1 = y_1^* + u$, $y_2 = y_2^* + v$, $y_3 = y_3^* + w$; where u , v , w are small perturbations.

Consider the following positive definite function for arbitrarily chosen positive constants D_1 , D_2 and D_3 :

$$V(t) = D_1 \left(u - y_1^* \log \left(1 + \frac{u}{y_1^*} \right) \right) + D_2 \left(v - y_2^* \log \left(1 + \frac{v}{y_2^*} \right) \right) + D_3 \left(w - x^* \log \left(1 + \frac{w}{x^*} \right) \right)$$

Then

$$\begin{aligned}
\frac{dV}{dt} &= D_1 u \left[1 - y_1^* - u - \frac{w_1 (x^* + w)}{1 + w_2 y_1 + w_3 y_2} \right] + D_2 v \left[w_4 (1 - y_2^* - v) - \frac{w_5 (x^* + w)}{1 + w_2 y_1 + w_3 y_2} \right] \\
&+ D_3 w \left(\frac{p_1 w_1 w_6 y_1 + p_2 w_5 w_7 y_2}{1 + w_2 y_1 + w_3 y_2} - C \right)
\end{aligned}$$

Rearranging and choosing arbitrary constants D_1 and D_2 as

$$D_2 = \alpha D_1; \text{ where } \alpha = \frac{w_1(w_7 w_5 p_2 - w_3 C)}{w_5(w_1 w_6 p_1 - w_2 C)} > 0, \text{ we get}$$

$$\frac{dV}{dt} = -\frac{D_2}{(1 + w_2 y_1 + w_3 y_2)} [m_1 u^2 + m_2 v^2 - C' uv]$$

$$\text{where } m_1 = (1 + w_2 y_1^* + w_3 y_2^*) - w_2 > 0,$$

$$m_2 = w_4 \alpha ((1 + w_2 y_1^* + w_3 y_2^*) - w_2) > 0, \quad C' = (w_3 + \alpha w_2 w_4) > 0$$

Therefore $\frac{dV}{dt} < 0$ provided

$$(w_3 + \alpha w_2 w_4)^2 < 4 \alpha w_4 ((1 + w_2 y_1^* + w_3 y_2^*) - w_2) ((1 + w_2 y_1^* + w_3 y_2^*) - w_3).$$

Further simplification yields

$$(w_3 - \alpha w_2 w_4)^2 < 4 \alpha w_4 ((1 + w_2 y_1^* + w_3 y_2^*) - w_2) ((1 + w_2 y_1^* + w_3 y_2^*) - w_3)$$

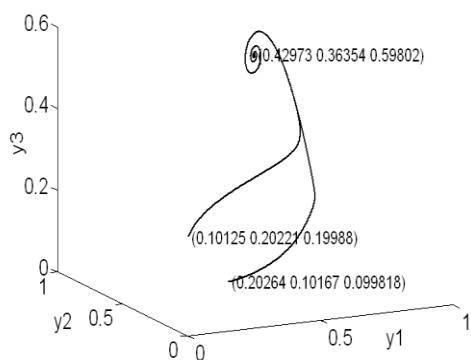
Thus, $\frac{dV}{dt}$ is negative definite when the condition (12) is satisfied. Therefore, V is a Liapunov function provided condition (12) is satisfied.

NUMERICAL SIMULATION:

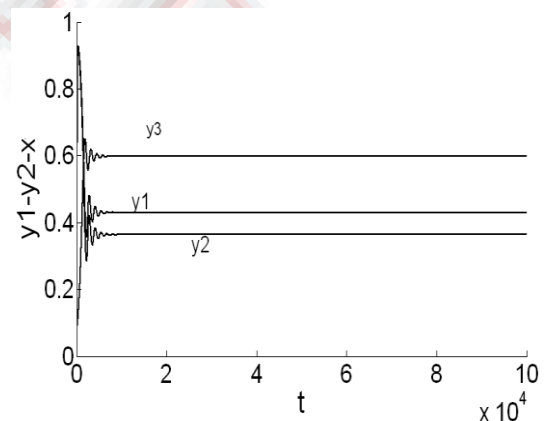
As the solution of the system is dissipative and bounded, the long time behavior of the solution is obtained as global attractor. For global dynamic behavior, numerical simulations of the underlying non-linear system are carried out. Consider the biological feasible set of parameters as

$$w_1 = 2.0, w_2 = 1.2, w_3 = 1.6, w_4 = 1.12, w_5 = 2.5, w_6 = 1.3, w_7 = 1.7, p_1 = 0.15, p_2 = 0.12, \quad (13)$$

The value of the expression $a_0 a_1 - a_2$ is $0.0030 > 0$ at $C = 0.156$. Fig. 1 (a) shows the phase space trajectories converging to the point $(0.42973, 0.36954, 0.59802)$ for the data set (34) at $C = 0.156$, starting with two different initial conditions. In other words, the nontrivial equilibrium point $(0.42973, 0.36954, 0.59802)$ is stable giving the persistence of the system for the given set of parameters. Fig 1(b) shows the time series for the data set (34) at $C = 0.156$.



(a)



(b)

Fig. : For the data set (34); (a) 3D behavior (b) time series

CONCLUSIONS:

In this model, separate dynamics of harvesting effort is considered. The local and global persistence of the harvested preys has been analyzed. Numerical simulation has been carried out for a set of data of parameters. The global stability is shown numerically as well as by phase diagram for a chosen set of data of parameters. Matlab is for numerical simulation and phase diagram. Global stability is justified by taking two set of initial values.

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